Equação de Convecção-Difusão-Reação

Achar

$u = u(x,y,t)$ tq

$$\frac{\partial u}{\partial t} - \nabla\cdot(K\nabla u) + \beta\cdot\nabla u + \sigma u = f \quad \text{em } \Omega\times(0,t_f)$$

$$u(x,y,0) = u_0(x) \quad \text{e} \quad u = g \text{ em } \partial\Omega$$

$\sigma \geq 0$, $\|K\| \neq 0$, $\nabla\cdot\beta = 0$, $f \in L^2(\Omega)$; $u : \Omega\times(0,t_f) \to \mathbb{R}$

Método de Galerkin

Achar $u_h \in X_h \subset H_0^1(\Omega)$ tq

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + B(u_h, v_h) = (f, v_h) \quad \forall v_h \in X_h \in H_0^1(\Omega)$$

ou seja,

Achar $u_h \in X_h$ tq

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h \, d\Omega + \int_\Omega \nabla u_h \cdot K \nabla v_h \, d\Omega + \int_\Omega \beta\cdot\nabla u_h \, v_h \, d\Omega + \int_\Omega \sigma u_h v_h \, d\Omega =$$

$$\int_\Omega f v_h \, d\Omega \quad \forall v_h \in X_h$$

Métodos Variacionais Estabilizados

Achar $u_h \in X_h$ tq

$$\underbrace{\left(\frac{\partial u_h}{\partial t}, v_h\right) + B(u_h, v_h)}_{\text{Galerkin}} + \underbrace{E(u_h, v_h)}_{\text{Termo de Perturbação}} = (f, v_h) \quad \forall v_h \in X_h$$

onde,

$$E(u_h, v_h) = \sum_{\Omega_e \in T_h} \int_{\Omega_e} (\mathcal{L}u_h - f)\, \tau_h\, P(v_h)\, d\Omega$$

$P(\cdot)$ é um operador aplicado no espaço das funções testes - caracteriza cada método estabilizado

τ_h é um parâmetro de estabilização.

$(\mathcal{L}u_h - f) = R(u_h)$ é o resíduo da equação diferencial.

SUPG (Streamline Upwind Petrov-Galerkin):

$$P(u_h) = \beta \cdot \nabla v_h$$

$$\mathcal{L}u_h - f = R(u_h) = \frac{\partial u_h}{\partial t} - \nabla \cdot (k \nabla u_h) + \beta \cdot \nabla u_h + r u_h - f$$

Ou seja,

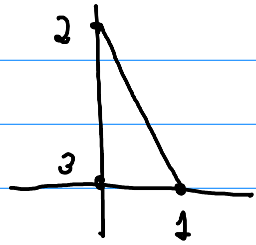
Achar $u_h \in X_h$ tq

$$\int_\Omega \frac{\partial u_h}{\partial t} v_h\, d\Omega + \int_\Omega \nabla u_h \cdot k \nabla v_h\, d\Omega + \int_\Omega \beta \cdot \nabla u_h\, v_h\, d\Omega + \int_\Omega r u_h v_h\, d\Omega +$$

$$\sum_{\Omega_e \in T_h} \tau_h \int_{\Omega_e} \frac{\partial u_h}{\partial t} \beta \cdot \nabla v_h\, d\Omega - \sum_{\Omega_e \in T_h} \tau_h \int_{\Omega_e} \nabla \cdot (k \nabla u_h) \beta \cdot \nabla v_h\, d\Omega + \sum_{\Omega_e \in T_h} \tau_h \int_{\Omega_e} \beta \cdot \nabla u_h\, \beta \cdot \nabla v_h\, d\Omega$$

$$+\sum_{\Omega_e \in T_h} \tau_h \int_{\Omega_e} r u_h \, \beta \cdot \nabla v_h \, d\Omega = \int_{\Omega} f v_h \, d\Omega + \sum_{\Omega_e \in T_h} \tau_h \int_{\Omega_e} f v_h \, \beta \cdot \nabla v_h \, d\Omega \qquad \forall v_h \in X_h$$

considerando Elemento triangular linear:

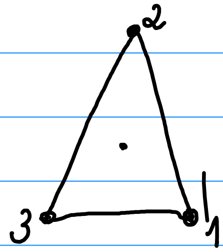


$$N_1(\xi,\eta) = \xi$$
$$N_2(\xi,\eta) = \eta$$
$$N_3(\xi,\eta) = 1-\xi-\eta$$

$$[\nabla N_i] = B = \frac{1}{2A^e}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$$

$$N = [N_1 \quad N_2 \quad N_3]$$

$$u_h^e = \sum_{i=1}^{3} u_i N_i$$



$$v_h^e = N_i$$

$$\frac{\partial u_h^e}{\partial t} = \sum_{i=1}^{3} \dot{u}_i N_i,$$ onde $\dot{u}_i$ é o valor nodal de $\frac{\partial u_h^e}{\partial t}$

$$f^e = \sum_{i=1}^{3} f_i N_i$$

A formulação SUPG resulta em um sistema de equações diferenciais ordinárias dado por:

$$M\dot{U} + KU = F$$

onde

$$M = \mathop{A}_{e=1}^{nel} (M_G^e + M_{PG}^e)$$

$$K = \mathop{A}_{e=1}^{nel} (D_G^e + D_{PG}^e + A_G^e + A_{PG}^e + R_G^e + R_{PG})$$

$$F = \mathop{A}_{e=1}^{nel} (F_G^e + F_{PG}^e)$$

A seguir associamos cada parcela da formulação a sua respectiva matriz local.

$$M_G^e : \int_{\Omega_e} \frac{\partial u_h}{\partial t} v_h \, d\Omega$$

$$M_{PG}^e : \int_{\Omega_e} \tau_h \frac{\partial u_h}{\partial t} \beta \cdot \nabla v_h \, d\Omega$$

$$D_G^e : \int_{\Omega_e} \nabla u_h \cdot k \nabla v_h \, d\Omega$$

$$D_{PG}^e : -\int_{\Omega_e} \tau_h \nabla \cdot (k \nabla u_h) \, \beta \cdot \nabla v_h \, d\Omega$$

$$A_G^e : \int_{\Omega_e} \beta \cdot \nabla u_h \, v_h \, d\Omega$$

$$A_{PG}^e : \int_{\Omega_e} \tau_h \beta \cdot \nabla u_h \, \beta \cdot \nabla v_h \, d\Omega$$

$$R_G^e : \int_{\Omega_e} r u_h v_h \, d\Omega$$

$$R_{PG}^e : \int_{\Omega_e} \tau_h r u_h \, \beta \cdot \nabla v_h \, d\Omega$$

$$F_G^e : \int_{\Omega_e} f v_h \, d\Omega$$

$$F_{PG}^e : \int_{\Omega_e} \tau_h f \, \beta \cdot \nabla v_h \, d\Omega$$

$$M_G^e = \int_{\Omega^e} N^T N \, d\Omega = \frac{A^e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$M_{PG}^e = \int_{\Omega^e} \tau_h B^T \beta N \, d\Omega = \tau_h B^T \beta \int_{\Omega^e} N \, d\Omega$$

$$M_{PG}^e = \tau_h \frac{1}{\cancel{2A^e}} \begin{bmatrix} y_{23} & x_{32} \\ y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix}^T \frac{\cancel{2A^e}}{6} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{\tau_h}{6} \begin{bmatrix} m_1 & m_1 & m_1 \\ m_2 & m_2 & m_2 \\ m_3 & m_3 & m_3 \end{bmatrix},$$

onde $m_1 = y_{23}\beta_x + x_{32}\beta_y$

$m_2 = y_{31}\beta_x + x_{13}\beta_y$

$m_3 = y_{12}\beta_x + x_{21}\beta_y$

$$D_G^e = \int_{\Omega_e} B^T K B \, d\Omega$$

$$= \frac{1}{2A^e} \begin{bmatrix} y_{23} & x_{32} \\ y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{bmatrix} \frac{1}{2\cancel{A^e}} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \cancel{A^e}$$

$$= \frac{1}{4A^e} \begin{bmatrix} -(d_{12}+d_{13}) & d_{12} & d_{13} \\ sim & -(d_{12}+d_{23}) & d_{23} \\ & & -(d_{13}+d_{23}) \end{bmatrix}$$

$d_{12} = y_{31}(k_{xx}y_{23} + k_{yx}x_{32}) + x_{13}(k_{xy}y_{23} + k_{yy}x_{32})$

$d_{13} = y_{12}(k_{xx}y_{23} + k_{yx}x_{32}) + x_{21}(k_{xy}y_{23} + k_{yy}x_{32})$

$d_{23} = y_{12}(k_{xx}y_{31} + k_{yx}x_{13}) + x_{21}(k_{xy}y_{31} + k_{yy}x_{13})$

$D_{PG}^e = 0$, pois $\nabla \cdot (K \nabla u_h) = 0$ para aproximação linear!

$$A_G^e = \int_{\Omega_e} N^T \beta^T B \, d\Omega = \int_{\Omega_e} N^T d\Omega \, \beta^T B$$

$$= \frac{\cancel{2A^e}}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \beta_x & \beta_y \end{bmatrix} \frac{1}{\cancel{2A^e}} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

onde, $a_1 = \beta_x y_{23} + \beta_y x_{32} = m_1$

$a_2 = \beta_x y_{31} + \beta_y x_{13} = m_2$

$a_3 = \beta_x y_{12} + \beta_y x_{21} = m_3$

A menos das constantes $A_G^e = (M_{PG}^e)^T$

$$A^e_{PG} = \tau_h \int_\Omega B^T \beta \beta^T B \, d\Omega \qquad \beta = \begin{bmatrix} \beta_{xx} & \beta_{xy} \\ \beta_{yx} & \beta_{yy} \end{bmatrix}$$

$$= \tau_h \frac{1}{2A^e} \begin{bmatrix} y_{23} & x_{32} \\ y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} \beta_x\beta_x & \beta_x\beta_y \\ \beta_y\beta_x & \beta_y\beta_y \end{bmatrix} \frac{1}{2\cancel{A}^e} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \cancel{A}^e$$

$$= \frac{\tau_h}{4A^e} \begin{bmatrix} -(a_{12}+a_{13}) & a_{12} & a_{13} \\ & -(a_{12}+a_{23}) & a_{23} \\ & & -(a_{13}+a_{23}) \end{bmatrix}$$

$$a_{12} = y_{31}(\beta_{xx} y_{23} + \beta_{yx} x_{32}) + x_{13}(\beta_{xy} y_{23} + \beta_{yy} x_{32})$$

$$a_{13} = y_{12}(\beta_{xx} y_{23} + \beta_{yx} x_{32}) + x_{21}(\beta_{xy} y_{23} + \beta_{yy} x_{32})$$

$$a_{23} = y_{12}(xx\, y_{31} + yx\, x_{13}) + x_{21}(\beta_{xy} y_{31} + \beta_{yy} x_{13})$$

$$R^e_G = \int_{\Omega^e} \sigma N^T N \, d\Omega$$

$$= \frac{\sigma A^e}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$R^e_{PG} = \int_{\Omega^e} \sigma \tau_h N^T \beta^T B \, d\Omega = \frac{\sigma \tau_h}{6} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

$$F^e_G = \int_{\Omega^e} N^T \sum_{i=1}^{3} f_i N_i \, d\Omega = \frac{A^e}{12} \begin{bmatrix} 2f_1 + f_2 + f_3 \\ f_1 + 2f_2 + f_3 \\ f_1 + f_2 + 2f_3 \end{bmatrix}$$

$$F^e_{PG} = \int_{\Omega^e} \tau_h \sum_{i=1}^{3} f_i N_i B^T \beta \, d\Omega = \frac{\tau_h}{\cancel{2}\cancel{A}^e} \begin{bmatrix} y_{23} & x_{32} \\ y_{31} & x_{13} \\ y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} \frac{\cancel{2}\cancel{A}^e}{6} (f_1 + f_2 + f_3)$$

$$= \frac{\tau_h}{6} (f_1 + f_2 + f_3) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$